

CONTACT-TIME BOUNDS FOR PAIRED GAIT ANGLES: SIGNED PHASE DIFFERENCES AND SAMPLED-SIGNAL ERROR

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ABSTRACT

Errors in the two contacts delimiting a gait cycle alter the angles compared at a common normalized phase. To determine the signed angular differences that persist under bounded contact-time corrections and sampled-signal error. Start- and end-contact intervals were projected through affine cycle normalization. Signal extrema on the resulting time intervals gave exact difference limits for independently selectable correction sets. Four trajectories were evaluated in 180 combinations of cycle duration and endpoint bounds; 216 sampled-signal combinations assessed linear interpolation with bounded sample perturbations. A four-event predictive box was calibrated from 199 exchangeable correction tuples. Opposite 10 ms contact-label offsets generated a nominal root-mean-square difference of 1.985° between identical trajectories. A 5° angular offset exceeded 2° for every admitted correction over 85.26%, 41.58% and 12.16% of phase at 10, 20 and 40 ms radii. Taylor bounds containing a curvature remainder enclosed the exact limits in all 144 smooth-trajectory combinations; first-order bounds omitted a limit in each combination. Their median mean-width ratio was 1.309. All 21,600 sampled-signal correction tuples satisfied the reconstruction-expanded limits at the evaluated phases. Across 200 calibration runs, mean joint coverage was 94.91%, with Monte Carlo standard error 0.12 percentage points. Contact-time intervals quantify the angular consequences of uncertain cycle boundaries. Shared corrections, local curvature and interpolation error determine the resulting limits. The retained phase fraction describes signed threshold exceedance under the stated measurement assumptions; participant-level treatment effects require a separate statistical analysis.

Keywords: gait analysis; joint kinematics; event timing; measurement uncertainty; interval propagation; signal reconstruction.

INTRODUCTION

An angular value at 40% of a gait cycle is evaluated at a clock time determined by the two contacts delimiting that cycle. Moving either contact changes this time even when the recorded trajectory is unchanged. Honert and Pataky demonstrated that segmentation and alignment events affect whole-trajectory statistical analyses of lower-limb biomechanics (Honert and Pataky, 2021). Alijanpour and Russell separated phase-duration differences from angular magnitude comparisons through gait-phase normalization (Alijanpour and Russell, 2024). Both studies establish the importance of the temporal correspondence used in gait analysis.

Contact detection introduces an additional uncertainty within a chosen event definition. Kinematic detection depends

on pathological gait pattern (Bruening and Ridge, 2014), while agreement with force-based events depends on the acquisition conditions (Hendershot et al., 2016). Marker-based detection during complex motor tasks affects stride-parameter calculation (Bonci et al., 2022). In video-based sports analysis, event-location error also changes angular performance parameters (Mundt et al., 2024). An error of 10 ms can produce different angular deviations on a steep segment and near a stationary point of the same trajectory. Cycle duration changes the proportion of phase occupied by that error.

Angular uncertainty can also originate upstream of contact detection. Myers and colleagues propagated measurement and parameter distributions through musculoskeletal simulations (Myers et al., 2015). Uchida and Seth quantified the consequences of marker-registration and model-scaling errors in inverse analyses (Uchida and Seth, 2022). Contact-time propagation addresses the temporal coordinate of an angular comparison. A bounded angular reconstruction error can be added separately when its maximum over the required recording interval is known.

This study derives minimum and maximum paired angular differences for signed corrections to four delimiting contacts. It examines whether a difference exceeds a specified magnitude at each phase for every admissible correction. The calculations distinguish shared contact errors from independently selectable errors and distinguish phasewise threshold retention from the minimum above-threshold fraction of one complete corrected cycle. Three paired contrasts isolate opposite contact-label offsets, a constant angular offset and a physical signal lag. Four explicitly defined trajectories, three matched cycle durations and three endpoint-bound profiles then quantify the effects of waveform shape and temporal scale. Linear interpolation and joint event calibration complete the analysis of the stated measurement assumptions.

CONTACT-TIME PROPAGATION

ANGULAR COORDINATES AND SIGNED CONTACT CORRECTIONS

Let $x_c(t)$ be a continuous scalar angular trajectory in degrees for condition $c \in \{A, B\}$, with clock time t in seconds. Angles must use a declared coordinate convention and, where necessary, an unwrapped representation. Nominal initial and subsequent ipsilateral contacts are s_c and e_c , with duration $T_c = e_c - s_c > 0$. The nominal sampling time at normalized phase $u \in [0, 1]$ is

$$t_c(u) = (1 - u)s_c + ue_c. \quad (1)$$

Corrections a_c and b_c to the two contacts are allowed within signed intervals. They need not be symmetric about zero:

$$a_c \in [a_c^-, a_c^+], \quad b_c \in [b_c^-, b_c^+], \quad h_c(u) = (1 - u)a_c + ub_c. \quad (2)$$

The four-event correction tuple is $\theta = (a_A, b_A, a_B, b_B)$. Under the product-box model, the attainable corrected sampling times at a fixed phase form

$$I_c(u) = [t_c(u) + (1 - u)a_c^- + ub_c^-, t_c(u) + (1 - u)a_c^+ + ub_c^+]. \quad (3)$$

Both coefficients in this linear projection are nonnegative. Consequently, the endpoints are attained and every intermediate time is attainable. For symmetric contact bounds $|a_c| \leq \eta_{sc}$ and $|b_c| \leq \eta_{ec}$, Eq. (3) reduces to $I_c(u) = [t_c(u) - r_c(u), t_c(u) + r_c(u)]$, where

$$r_c(u) = (1 - u)\eta_{sc} + u\eta_{ec}. \quad (4)$$

Equal endpoint bounds yield a constant radius in clock time; unequal bounds change the radius across phase. Its fraction of the nominal cycle is $r_c(u)/T_c$. A 10 ms radius occupies 1.25% of a 0.8 s cycle and 0.83% of a 1.2 s cycle. These phase proportions follow directly from the specified cycle durations.

Every pair in the unrestricted contact rectangle preserves event order if

$$T_c + b_c^- - a_c^+ > 0. \quad (5)$$

The recording must contain the entire projected time buffer. Otherwise the admissible set must be restricted, or extrapolation must be declared as an additional model. Missing, extra or incorrectly paired contacts are segmentation failures outside this endpoint-correction model. The formulation presumes that the correct two physical events have been identified.

EXACT POINTWISE DIFFERENCE BOUNDS

Define the corrected difference, and the extrema of each signal on its projected interval, by

$$D_\theta(u) = x_A(t_A(u) + h_A(u)) - x_B(t_B(u) + h_B(u)), \quad \ell_c(u) = \min_{t \in I_c(u)} x_c(t), \quad q_c(u) = \max_{t \in I_c(u)} x_c(t). \quad (6)$$

Continuity on each compact interval ensures that these extrema exist. If the admissible corrections are the product of the two contact rectangles, the exact pointwise limits are

$$L(u) = \ell_A(u) - q_B(u), \quad U(u) = q_A(u) - \ell_B(u). \quad (7)$$

Proposition 1. Under these conditions, $L(u)$ and $U(u)$ are the minimum and maximum of $D_\theta(u)$ at each phase. Every admissible fixed tuple satisfies $L(u) \leq D_\theta(u) \leq U(u)$ throughout the cycle.

Proof. Each signal reaches every time in its projected interval. At a fixed phase, a time attaining the minimum of A and a separately admissible time attaining the maximum of B can be selected together under the product-set assumption. This yields the lower limit. The reversed selections yield the upper limit. Every fixed tuple projects into both intervals at every phase, establishing simultaneous trajectory containment. The requirement is geometric separability of the correction set, not probabilistic independence. $\square$

The tuple attaining a boundary at one phase can differ from that attaining it elsewhere. The envelope is pointwise sharp but generally does not constitute one realizable corrected trajectory. Furthermore, when signal uncertainty is added later, the expanded limits are conservative rather than necessarily sharp for a joint event-and-signal model.

DEPENDENCE BETWEEN EVENT CORRECTIONS

For a compact joint correction set Θ that contains synchronization or event-sharing constraints, the appropriate limits are

$$L_\Theta(u) = \min_{\theta \in \Theta} D_\theta(u), \quad U_\Theta(u) = \max_{\theta \in \Theta} D_\theta(u). \quad (8)$$

Eq. (7) encloses these limits when its box contains Θ , but may be wider. A simple exact example has identical nominal windows, $x_A(t) = g(t) + c$, $x_B(t) = g(t)$, and shared corrections $a_A = a_B$, $b_A = b_B$. The corrected difference is then identically c , regardless of the common correction radius. Separately varying corrections would introduce angular variation excluded by that measurement model.

Adjacent strides share a physical boundary event. A joint analysis must assign one correction to that event rather than permit incompatible corrections in neighboring strides. Known common synchronization components should likewise be represented explicitly. If dependence cannot be justified, a containing rectangle remains a conservative sensitivity model, provided it is identified as such. No correlation coefficient is needed for the deterministic calculation, and an estimated correlation alone does not define a bounded joint set.

SIGNED QUALIFICATION AND THE WHOLE-CYCLE DISTINCTION

Choose a cutoff $m \geq 0$ in degrees before interpretation. A positive difference is qualified at phase u if $L(u) > m$; a negative difference is qualified if $U(u) < -m$. A nominal above-cutoff difference is timing-sensitive when neither condition holds. Define

$$R_m = 100 \int_0^1 \mathbf{1}\{L(u) > m \text{ or } U(u) < -m\} du. \quad (9)$$

This is the percentage of phase positions satisfying a signed threshold for every admitted tuple. It is not generally the minimum whole-cycle above-threshold fraction obtainable from one fixed tuple. If

$$F_m(\theta) = 100 \int_0^1 \mathbf{1}\{|D_\theta(u)| > m\} du, \quad R_m \leq \inf_{\theta \in \Theta} F_m(\theta), \quad (10)$$

the inequality follows because qualification implies the threshold for every tuple at that phase. It can be strict: the tuple responsible for loss of qualification can vary across phase. A zero value of R_m therefore identifies absence of a universally qualified phase position, not absence of a movement difference.

The cutoff may represent a justified measurement tolerance or scientific magnitude of interest. The 2° cutoff used in the calculations below is illustrative and is not asserted to be a minimum clinically important difference. Neither R_m nor the envelope width is a participant response proportion, an effect probability, or a family-wise confidence level.

DERIVATIVE AND RECONSTRUCTION ENCLOSURES

For a twice continuously differentiable signal, let K_c bound $|x_c''|$ uniformly on the necessary buffer, and let D_0 be the nominal difference. For signed intervals, use $\bar{r}_c(u) = \max\{|(1-u)a_c^- + ub_c^-|, |(1-u)a_c^+ + ub_c^+|\}$. Taylor's theorem gives

$$|D_\theta(u) - D_0(u)| \leq B(u) + Q(u), \quad B(u) = |x_A'(t_A(u))|\bar{r}_A(u) + |x_B'(t_B(u))|\bar{r}_B(u), \quad Q(u) = \frac{1}{2}\{K_A\bar{r}_A(u)^2 + K_B\bar{r}_B(u)^2\}. \quad (11)$$

The remainder matters near stationary points, where the local slope can vanish despite a nonzero angular range. Without Q , the expression is a first-order approximation rather than a guaranteed enclosure. Bounds estimated solely from noisy local derivatives provide no mathematical guarantee. For a nonsmooth interpolant, use its interval extrema, or a justified Lipschitz constant H_c to replace each signal allowance by $H_c \bar{r}_c$.

Suppose a reconstructed signal $\tilde{x}_c$ differs from the target signal by at most ϵ_c uniformly on the buffer. Limits computed from the reconstruction can be expanded as

$$L_{\text{final}} = \tilde{L} - \epsilon_A - \epsilon_B, \quad U_{\text{final}} = \tilde{U} + \epsilon_A + \epsilon_B. \quad (12)$$

This statement requires a uniform allowance over every admitted time, not merely an average reconstruction error. Independence between timing and angular errors is unnecessary for this containing bound. For linear interpolation of a C^2 target sampled with maximum spacing Δt , and sample perturbations bounded by ν_c , a sufficient allowance is

$$\epsilon_c = \nu_c + \frac{K_c(\Delta t)^2}{8}. \quad (13)$$

The interpolation remainder gives the second term. The interpolated perturbation is a convex combination of adjacent sample errors, so its magnitude cannot exceed ν_c . The bound remains conditional on the target's smoothness and curvature allowance. A high sampling rate, visually smooth trace or pose-estimation confidence score does not establish these assumptions.

JOINT CALIBRATION OF FOUR CONTACT CORRECTIONS

A deterministic correction bound must contain the relevant event error. The finite range of a calibration sample alone does not establish such a bound. For a predictive set with marginal coverage, let $E_i \in \mathbb{R}^4$ be signed event-correction tuples, let positive scales w_j be fixed before calibration, and assume that the n calibration tuples and one future tuple are exchangeable. A maximum standardized score accounts for all four events:

$$S_i = \max_{1 \leq j \leq 4} |E_{ij}|/w_j, \quad k = \lceil (n+1)(1-\alpha) \rceil, \quad q = S_{(k)}. \quad (14)$$

If $k > n$, set $q = +\infty$. Otherwise the box $\Theta_q = \{\theta : |\theta_j| \leq qw_j\}$ satisfies

$$\Pr\{E_{n+1} \in \Theta_q\} \geq 1 - \alpha. \quad (15)$$

The order-statistic construction uses the finite-sample marginal coverage principle of conformal prediction (Lei et al., 2018). For continuous scores, the future score's rank among $n+1$ exchangeable values is uniform, giving probability $k/(n+1)$ of being no larger than the calibration order statistic. Ties make the non-strict inclusion conservative. Within-tuple dependence is permitted. The guarantee averages over calibration and future tuples; it is not conditional coverage for every realized calibration sample.

Event-tuple inclusion implies whole-trajectory containment through Proposition 1, provided the signal and event-order requirements also hold. Multiple future cycles do not automatically receive simultaneous $1 - \alpha$ coverage. Consecutive strides, population changes or altered contact definitions can undermine exchangeability. The scales must be prespecified or estimated using independent training data. Calibration against an operational reference also retains uncertainty in that reference's threshold, filtering and synchronization.

NUMERICAL METHODS

FOUR ANGULAR TRAJECTORIES AND THREE PAIRED CONTRASTS

The calculations used synthetic records with fixed angular functions, correction intervals and evaluation rules, consistent with explicit simulation-design specification (Morris et al., 2019). Angles were generated from the functions below, without participant measurements. The three smooth phase functions were

$$\begin{cases} q_{F1}(v) = 25 - 20 \cos(2\pi v) + 5 \sin(4\pi v), \\ q_{F2}(v) = 20 - 15 \cos(2\pi v) + 6 \sin(4\pi v) + 3 \cos(6\pi v), \\ q_{F3}(v) = 18 - 10 \cos(2\pi v) + 4 \sin(6\pi v) + 2 \cos(10\pi v). \end{cases} \quad (16)$$

Each function has unit period in normalized phase. The fourth function, $P1$, is the periodic linear interpolant through the following paired values:

$$\begin{cases} \text{Phase knots: } 0, 0.1, 0.2, 0.4, 0.6, 0.75, 0.9, 1; \\ \text{Angles in degrees: } 5, 15, 7, 10, 42, 25, 8, 5. \end{cases} \quad (17)$$

$F1$ contains the first and second harmonics, $F2$ adds a third harmonic, and $F3$ includes a fifth harmonic. Their sufficient phase-curvature bounds are $160\pi^2$, $264\pi^2$ and $384\pi^2$ degrees per unit phase squared, respectively. $P1$ has slope discontinuities at its interior knots. These functions specify the angular records used in the calculations.

The three paired contrasts used $g(t) = q_{F1}(t)$ and one-second cycles. The contact-label contrast compared identical records $x_A = x_B = g$, with nominal windows $[-0.010, 0.990]$ s and $[0.010, 1.010]$ s; both generating windows were $[0, 1]$ s. The constant-offset contrast used $x_A = g + 5$ and $x_B = g$ on $[0, 1]$ s. The signal-lag contrast used $x_A = g$ and $x_B(t) = g(t - 0.030)$ on the same nominal windows. The 30 ms displacement belongs to the signal definition, whereas the opposite 10 ms offsets belong to the contact labels.

Symmetric contact radii were 10, 20 and 40 ms. The constant-offset and signal-lag contrasts also used a 5 ms radius. The contact-label contrast required a radius of at least 10 ms to contain its specified corrections. The constant-offset calculation evaluated cutoffs $m = 0, 1, 2, 3, 4, 5$ degrees.

CYCLE DURATION AND ENDPOINT-BOUND COMBINATIONS

For each template, the constant-offset contrast used $x_A(t) = q(t/T_A) + 5$ and $x_B(t) = q(t/T_B)$, with nominal windows $[0, T_A]$ and $[0, T_B]$. Matched durations were 0.8, 1.0 and 1.2 s. Endpoint-bound profiles were balanced (ρ, ρ), end-dominant ($0.5\rho, 2\rho$) and start-dominant ($2\rho, 0.5\rho$), applied in both conditions, with $\rho = 5, 10, 20, 40$ ms. This produced $4 \times 3 \times 3 \times 4 = 144$ settings. The three smooth templates were additionally tested with $T_A = 0.8$ s and $T_B = 1.2$ s under all profiles and radii, producing 36 further settings and 180 in total. The nominal normalized difference remained 5° in every setting.

A signed-asymmetric example used $F2$, durations 0.8 and 1.2 s, and correction intervals $[-6, 14]$, $[-22, 8]$, $[-12, 8]$ and $[-10, 20]$ ms for (a_A, b_A, a_B, b_B) . A separate shared-error example imposed identical corrections on the matched $F1$ offset contrast and compared its exact constant difference with the containing product-box result. The signed intervals and equality constraints were fixed inputs to the angular calculation.

SAMPLING, ANGULAR PERTURBATIONS AND EVENT CALIBRATION

The reconstruction assessment crossed three smooth templates, three matched durations, four sampling rates (25, 50, 100 and 200 Hz), three sample-error bounds ($\nu = 0, 0.25, 0.50$ degrees), and two event radii (10 and 40 ms), giving 216 settings. Sample times began at -0.12 s and continued beyond the cycle's end with the same buffer. Independent sample perturbations were drawn uniformly within $[-\nu, \nu]$ for each condition. Every sample perturbation therefore satisfied the specified angular bound. Interval extrema of the resulting linear interpolants were expanded by Eq. (13) for both signals.

The synthetic calibration demonstration used $n = 199$, $\alpha = 0.05$ and prespecified scales $w = (6, 10, 8, 12)$ ms. Corrections were generated as

$$E_{ij} = w_j(0.6Z_i + 0.8Z_{ij}), \quad (18)$$

where the common and event-specific terms were independent standard normal variables. Thus each standardized coordinate had unit variance and pairwise correlation 0.36. In each of 200 independent replications, one calibration sample and 1,000 future tuples were generated. Mean future-tuple coverage and its Monte Carlo standard error were calculated across replication-level proportions. The normal distribution has unbounded support, so the calibrated box has marginal predictive coverage rather than deterministic coverage of every possible tuple.

EXTREMUM SEARCH, CONVERGENCE AND CHECKS

For $F1$, the stationary phases are 0.5596408373123332 and 0.9403591626876668; interval extrema are obtained from endpoints and every contained periodic stationary point. For $F2$ and $F3$, derivative roots were enumerated through the associated Laurent polynomial in $z = \exp(2\pi i\nu)$ and independently compared with sign-changing roots on a 20,000-subinterval phase grid. Roots on the unit circle were retained within 10^{-7} and checked for a derivative residual below 10^{-6} degrees per unit phase. The selected templates had 2, 2 and 10 stationary points per period, respectively. For $P1$ and the sampled-signal linear interpolants, endpoints and all contained knots were evaluated. The analytical extremum prescription is exact; floating-point evaluations are subject to the stated tolerances rather than formally certified rounding bounds.

Threshold domains for the three paired contrasts were obtained by locating crossings on 10,000 subintervals and refining sign-changing roots with Brent iteration to 10^{-13} in normalized phase. Nominal, positive and negative domains were checked at 20,000 and 40,000 subintervals. RMS values for the three paired contrasts used trapezoidal integration at 100,001 phase positions. Factorial fractions used 40,000 midpoint positions and were compared with 20,000 positions; phase-mean widths and derivative limits used 2,001 positions. Reconstruction fractions used 2,000 midpoint positions.

Complete four-event tuples, rather than independently changing errors at each phase, were used for containment checks. There were 1,000 tuples per paired contrast at the 10 ms radius and 2,001 phases; 100 tuples per factorial setting and 501 phases; 100 signed-asymmetric tuples; and 100 tuples per reconstruction setting and 201 phases. The paired-contrast, duration-bound, reconstruction and calibration seeds were 94173, 202611, 202612 and 202613. Numerical containment tolerance was 10^{-9} degrees. These finite checks supplement the proofs and do not establish probability coverage for gait data.

Taylor conservatism was summarized by the ratio of phase-mean Taylor width to phase-mean interval-extremum width. A first-order failure was recorded when $[D_0 - B, D_0 + B]$ omitted either interval limit at a checked phase. A separate 16-corner calculation at the second stationary phase of $F1$ and a 20 ms radius tested whether event-box corners suffice for nonlinear signals.

RESULTS

CONTACT-LABEL OFFSETS, ANGULAR OFFSETS AND SIGNAL LAG

The identical-record contact-label contrast produced RMS 1.984828° , with a nominal 2° above-cutoff domain of 23.3143%. Corrections of +10 ms to both A contacts and -10 ms to both B contacts restored the identical true windows, giving zero difference throughout phase. The 10 ms envelope therefore included zero everywhere. Table 1 gives RMS differences and phase fractions for the three paired contrasts.

TABLE 1. THREE PAIRED CONTRASTS GENERATED FROM $F1$. THE NOMINAL PHASE FRACTION SATISFIES $|D_0| > 2^\circ$; THE RETAINED FRACTION SATISFIES EQ. (9). RMS IS CALCULATED OVER THE NOMINAL ONE-SECOND CYCLE.

Contrast	Radius (ms)	RMS ($^\circ$)	Nominal (%)	Retained (%)
Opposite contact offsets	10	1.985	23.31	0.00
	20	1.985	23.31	0.00
	40	1.985	23.31	0.00
Angular offset	10	5.000	100.00	85.26
	20	5.000	100.00	41.58
	40	5.000	100.00	12.16
30 ms signal lag	10	2.973	69.28	0.00
	20	2.973	69.28	0.00
	40	2.973	69.28	0.00

For the 5° offset at 10 ms, the minimum lower limit was 1.235047° . The sign remained positive throughout phase, although qualification above 2° failed over 14.7427%. At 20 and 40 ms, positive sign was no longer guaranteed everywhere. A 5 ms radius retained the whole phase domain above 2° . Table 2 shows that numerical qualification also depends strongly on the selected cutoff. At 10 ms, changing the cutoff from 2° to 4° reduced the retained fraction from 85.26% to 16.41%.

TABLE 2. CUTOFF SENSITIVITY FOR THE $F1$ CONSTANT-OFFSET CONTRAST WITH BALANCED ENDPOINT BOUNDS AND A ONE-SECOND CYCLE. ENTRIES ARE R_m IN PERCENT. STRICT INEQUALITIES ARE USED, INCLUDING AT $m = 5^\circ$.

Radius (ms)	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	100.00	100.00	85.26	76.69	16.41	0.00
20	80.74	76.72	41.58	16.51	7.99	0.00
40	21.78	16.89	12.16	6.79	0.00	0.00

The imposed physical lag produced RMS 2.973329° and a nominal above-cutoff domain of 69.2772%. At 10 ms, no phase retained a signed magnitude above 2° for every admitted tuple. The physical signal still contained its imposed 30 ms lag. At a 5 ms radius, 23.3143% of phase retained a negative difference below -2° . Thus loss of qualification cannot by itself distinguish an event artefact from a genuine movement-timing difference.

DURATION, SHAPE AND DEPENDENCE

All 18,000 factorial event tuples and the 100 signed-asymmetric tuples were contained at the checked phase positions. Table 3 demonstrates that the same clock-time budget produces different phase fractions as duration and shape change. At balanced 10 ms bounds, $P1$ retained 81.74% for a one-second cycle but 100.00% for 1.2 s. This variation follows from local slopes, knots and the fraction of the cycle occupied by the timing radius; it is not evidence about between-participant variability.

TABLE 3. SELECTED FACTORIAL RESULTS FOR THE 5° OFFSET AND $m = 2^\circ$. BALANCED COLUMNS USE MATCHED CYCLE DURATIONS. THE LAST COLUMN USES A ONE-SECOND CYCLE WITH START/END RADII OF 5/20 MS. FRACTIONS USE A NUMERICAL MIDPOINT GRID.

Template	0.8 s (%)	1.0 s (%)	1.2 s (%)	End-dominant (%)
<i>F1</i>	79.85	85.26	93.27	75.63
<i>F2</i>	83.42	86.29	89.56	76.41
<i>F3</i>	83.55	88.12	94.98	78.20
<i>P1</i>	81.19	81.74	100.00	52.87

For the signed-asymmetric *F2* intervals, $R_2 = 80.435\%$. With matched *F1* cycles and shared corrections, $D_\theta(u) = 5^\circ$ at every phase, giving $R_2 = 100\%$. Allowing the same 20 ms endpoint corrections to vary separately between conditions gave $R_2 = 41.58\%$. Thus the equality constraints changed retention by 58.42 percentage points without changing the individual correction radii. Figure 1 traces the radius dependence for the four one-second trajectories.

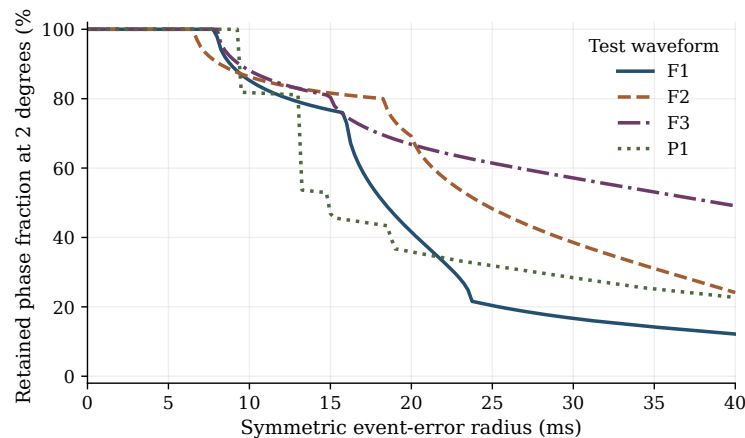


FIGURE 1. RETAINED PHASE FRACTION FOR A 5° ANGULAR OFFSET AS BALANCED TIMING RADII INCREASE. ALL TEMPLATES HAVE ONE-SECOND CYCLES AND A 2° CUTOFF. EACH CURVE IS CALCULATED FROM SYNTHETIC CONTINUOUS OR PIECEWISE-LINEAR RECORDS, USING 20,000 PHASE MIDPOINTS AT 161 RADII. THE CURVES DESCRIBE THE ADMITTED MEASUREMENT SET, NOT DETECTION-ERROR PROBABILITIES.

CURVATURE, CORNER EXTREMA AND NUMERICAL RESOLUTION

Taylor enclosures contained the interval limits in all 144 smooth-signal settings. The ratio of mean Taylor width to mean interval-extremum width ranged from 1.037 to 3.602, with median 1.309 (Figure 2). First-order limits without the remainder omitted at least one valid interval boundary in every setting. At those phases, the first-order calculation omitted angular values admitted by the specified event intervals. Lipschitz limits contained the *P1* envelopes in its 36 settings.

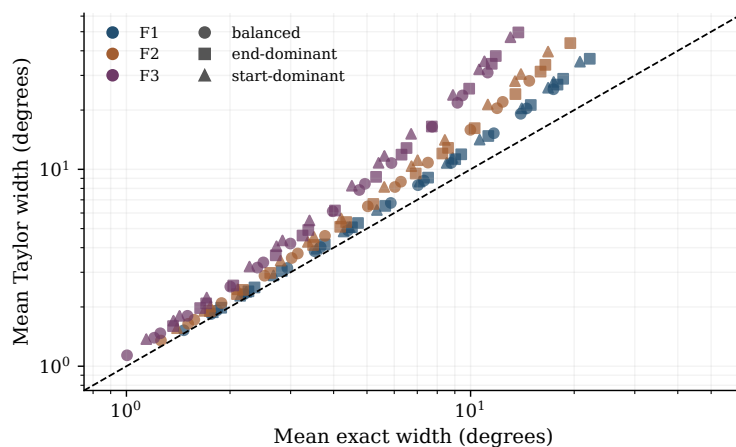


FIGURE 2. CONSERVATISM OF THE DERIVATIVE ENCLOSURE ACROSS 144 SMOOTH-SIGNAL SETTINGS. BOTH AXES ARE LOGARITHMIC AND SHOW WIDTHS AVERAGED OVER NORMALIZED PHASE. COLOR IDENTIFIES THE TEMPLATE; SYMBOL IDENTIFIES THE ENDPOINT-BOUND PROFILE. THE DASHED LINE DENOTES EQUAL WIDTH. OVERLAPPING POINTS CAN REPRESENT DIFFERENT DURATION OR BUDGET SETTINGS. ALL PLOTTED TAYLOR WIDTHS INCLUDE THE CURVATURE REMAINDER.

At phase 0.9403591627 in the 20 ms $F1$ offset example, interval extrema gave the difference range $[4.738990, 5.261010]^\circ$. The 16 event-box corners gave only $[4.931019, 5.068981]^\circ$. An admissible interior signal minimum caused the omitted values (Figure 3). A nonlinear signal therefore cannot generally be optimized by evaluating correction-box corners alone.

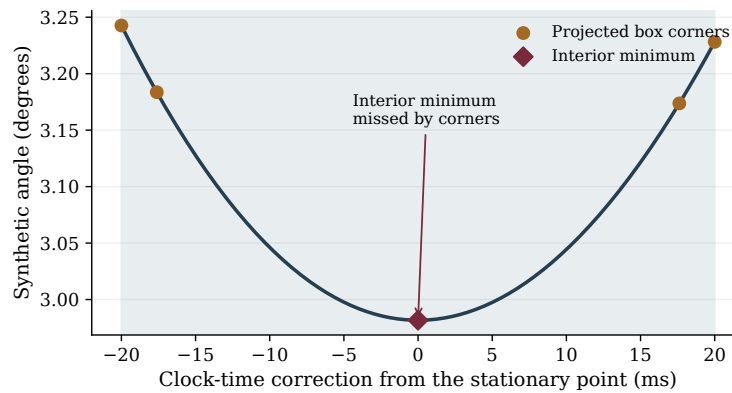


FIGURE 3. LOCAL SIGNAL SEARCH UNDERLYING THE 16-CORNER COUNTEREXAMPLE. THE INTERVAL IS CENTERED ON THE SECOND STATIONARY PHASE OF $F1$, WITH A 20 MS CORRECTION RADIUS. FOUR PROJECTED CONTACT-BOX CORNER TIMES ARE MARKED, WHILE THE DIAMOND IDENTIFIES A VALID INTERIOR MINIMUM. DIFFERENCE EXTREMA REQUIRE SELECTING SIGNAL EXTREMA IN THE TWO CONDITIONS; THE SIGNAL MINIMUM SHOWN HERE IS MISSED BY ALL CORNER PROJECTIONS.

The largest change in nominal and signed fractions for the three paired contrasts was below 3×10^{-12} percentage points. Refining the duration-bound calculations from 20,000 to 40,000 phase midpoints changed their fractions by at most 0.020 percentage points. Table 3 therefore gives approximate phase percentages. Finite-grid agreement cannot guarantee detection of every threshold crossing in an arbitrary measured record.

RECONSTRUCTION AND EVENT-SET CALIBRATION

All 21,600 reconstruction-assessment tuples were contained at the checked phases after the paired signal allowance was added. The largest observed single-signal reconstruction error divided by its sufficient allowance was 0.988. Table 4 gives selected $F1$ results with one-second cycles, $\nu = 0.25^\circ$ and a 10 ms event radius. The observed error concerns one reconstructed signal; the allowance is the sum for both signals. It is therefore not an empirical paired-error estimate.

TABLE 4. SELECTED NOISY-RECONSTRUCTION RESULTS. THE PAIRED ALLOWANCE IS $2\nu + 2K/(8f^2)$, WITH $K = 160\pi^2$ DEGREES PER SECOND SQUARED FOR $F1$. OBSERVED VALUES ARE MAXIMA ACROSS THE TWO INDIVIDUAL RECONSTRUCTED RECORDS, EVALUATED ON A DENSE CLOCK-TIME GRID.

Sampling rate (Hz)	Paired allowance ($^\circ$)	Observed single-signal error ($^\circ$)
200	0.509870	0.248803
100	0.539478	0.248448
50	0.657914	0.249461
25	1.131655	0.471106

For the calibration example, $k = 190$. The first realization yielded $q = 2.628342$ and contact radii of 15.77, 26.28, 21.03 and 31.54 ms. Across 200 replications, mean joint future-tuple coverage was 94.9055%, with Monte Carlo standard error 0.1232 percentage points. Realized validation proportions ranged from 90.10% to 98.80%. This variation illustrates why marginal coverage over calibration samples must not be described as a 95% conditional guarantee for each selected box. These coverage values describe the four-dimensional normal generator in Eq. (18).

DISCUSSION

ANGULAR CONSEQUENCES OF CONTACT-TIME UNCERTAINTY

The opposite contact-label offsets produced a 1.985° RMS difference despite identical generating signals. A correction tuple restoring both windows to $[0, 1]$ s yielded zero difference at every phase. The envelope correctly admitted that tuple. For the separate 30 ms signal lag, no phase retained a signed difference above 2° at a 10 ms correction radius, although the lag remained part of the generating function. Thus threshold loss identifies compatibility with the allowed timing corrections; it does not identify the physical cause of a difference. Contact timing and phase duration should also be analyzed when they are movement outcomes.

For the 5° offset, widening a balanced contact radius from 10 to 20 ms reduced R_2 from 85.26% to 41.58%. The same 10 ms clock-time radius occupied 1.25% of a 0.8 s cycle and 0.83% of a 1.2 s cycle. Duration therefore changes the angular range searched around each normalized phase. Template $P1$ retained 81.74% at 1.0 s and 100.00% at 1.2 s because its piecewise slopes and knots interact with that range. These values describe the specified functions rather than variation among participants.

The shared-error calculation isolates the importance of event constraints. When both conditions receive identical corrections, the common function cancels and the 5° difference is invariant. Permitting separate corrections discards that cancellation and broadens the limits. Shared stride boundaries and synchronization offsets should therefore use the same correction variable wherever the physical event is common. A rectangular set is conservative when it contains a more restricted joint set, but its wider limits can eliminate threshold retention.

The 16-corner calculation failed because a stationary point lay inside the projected interval. Evaluating contact-box corners alone gave $[4.931019, 5.068981]^\circ$, whereas the full interval search gave $[4.738990, 5.261010]^\circ$. The derivative enclosure avoids this omission when the curvature remainder is retained. Its median mean-width ratio of 1.309 quantifies the extra width across the 144 smooth-function combinations. Direct interval extrema are preferable when all stationary points or interpolation knots can be enumerated.

CALIBRATION AND ANGULAR RECONSTRUCTION

The maximum standardized score calibrates four event errors jointly. At $n = 199$ and $\alpha = 0.05$, the 190th calibration score defines a box with at least 95% marginal coverage for one exchangeable future tuple. The first generated calibration sample gave radii of 15.77, 26.28, 21.03 and 31.54 ms. Coverage across the 200 runs averaged 94.91%; individual validation proportions ranged from 90.10% to 98.80%. These results accord with a marginal guarantee averaged over calibration samples, rather than a fixed 95% conditional coverage for each realized box.

Measured calibration errors depend on task, population, reference event and synchronization. Romijnders and colleagues examined curved walking and turning in older adults and people with Parkinson's disease (Romijnders et al., 2021). Munoz-Larrosa and colleagues evaluated amputee gait, terrain and algorithm effects (Munoz-Larrosa et al., 2024). Prasanth and colleagues reviewed wearable real-time detection methods across application settings (Prasanth et al., 2021). A calibration set drawn under one event definition or walking task therefore requires justification before use under another. Within-tuple dependence is allowed in the rank calculation; exchangeability across calibration and future tuples remains essential. Consecutive dependent strides cannot be treated as exchangeable independent observations without examining the sampling design.

The sampled-signal calculations used known perturbation bounds and curvature constants. For $F1$ at 100 Hz with $\nu = 0.25^\circ$, the paired allowance was 0.539478° ; at 25 Hz it was 1.131655° . The maximum observed error of an individual reconstructed record was 0.248448° and 0.471106° , respectively. The paired allowance bounds the sum of two individual errors and should be distinguished from those single-record maxima. Measured angular records require a uniform error allowance over the entire contact-correction buffer. An average error or an acquisition confidence score alone cannot supply that allowance.

Clock-time records should include the full projected interval, the nominal contact times and identifiers for shared events. For a linear interpolant, its extrema are found at the interval endpoints and all enclosed knots. For a smooth function, the candidate set also includes every enclosed stationary point. A direct scan at P phase positions over N knots has order $O(PN)$ cost per condition. The Fourier calculations used a finite stationary-point list, while constrained contact sets require optimization over Eq. (8). Random correction tuples verify selected values but cannot establish the global extrema.

PARTICIPANT WEIGHTING AND STATISTICAL INTERPRETATION

The calculations concern scalar angular records and their contact corrections. In a paired participant study, a fixed nonnegative weighting rule can aggregate record-level envelopes. For weights w_i summing to one,

$$\sum_i w_i L_i(u) \leq \sum_i w_i D_{i,\theta_i}(u) \leq \sum_i w_i U_i(u). \quad (19)$$

Shared events can restrict the admissible combinations and make these weighted limits conservative. Equal participant weights prevent a larger number of strides from giving one participant greater influence. Weights selected after examining angular outcomes or contact corrections require a separate derivation.

Between-participant sampling uncertainty is distinct from the contact-time and reconstruction bounds. Whole-trajectory statistical methods address dependence and multiple testing along a continuum (Pataky et al., 2013, 2015); a prespecified region of interest can restrict the inference domain (Pataky et al., 2016). The angular envelope can accompany such

analyses, but its width measures sensitivity to the stated event and signal errors. It is not a confidence interval for a population treatment effect.

All numerical trajectories were synthetic. The smoothness constants, angular offsets, timing lag and sample-error bounds were fixed by construction. The results establish the behavior of the equations under these inputs, not detector accuracy in clinical gait or a therapeutic response. Missing contacts, incorrectly paired events, coordinate errors and unconstrained model-registration errors fall outside the four-contact calculation. Clinical use requires angular and event measurements whose error properties match the stated assumptions.

CONCLUSION

Signed corrections to the two contacts in each condition produce exact pointwise angular-difference limits when the correction sets are independently selectable. For the one-second $F1$ trajectory with a 5° offset, threshold retention at 2° decreased from 85.26% to 12.16% as the balanced radius increased from 10 to 40 ms. Shared corrections preserved the full offset, while separate corrections broadened the admissible differences. Curvature terms and uniform interpolation allowances were necessary to enclose the tested values. The retained phase fraction quantifies which signed differences persist under the specified timing and angular bounds; population inference remains a separate analysis.

DECLARATIONS

AUTHOR CONTRIBUTIONS

All authors contributed equally and all authors read and approved the final version of this manuscript.

CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest related to this study.

DATA AVAILABILITY

Not Applicable.

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